18.152 Final assignment

due May 12nd 10:00 pm

1. (10 points) Determine whether the following statements are true or false, and briefly verify your answer.

(A) Suppose that a smooth function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ satisfies $u_t = 2u_x$. Then, the solution u is uniquely determined by its initial data $u(x,0) \in C^\infty(\mathbb{R})$.

(B) An entire smooth function $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ satisfies the heat equation $u_t = \Delta u$ at all $(x,t) \in \mathbb{R}^n \times [0,T)$. Suppose that $u(x,0)$ is compactly supported. Then, for each $t > 0$, $u(x,t)$ is also compactly supported.

(C) An entire smooth function $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ satisfies the wave equation $u_{tt} = \Delta u$ at all $(x,t) \in \mathbb{R}^n \times [0,T)$. Suppose that $u(x,0)$ is compactly supported. Then, for each $t > 0$, $u(x,t)$ is also compactly supported.

(D) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Suppose that $u, v \in C^\infty(\Omega)$ are Dirichlet Laplace eigenfunctions such that the set $\{u \neq v\}$ has positive measure and $\|u\|_{L^2(\Omega)} \neq \|v\|_{L^2(\Omega)}$. Then, the following holds

$$\int_\Omega u(x)v(x)dx = 0. \tag{1}$$

(E) Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary $\partial\Omega$. Given $f, g \in C^\infty(\overline{\Omega})$, the elliptic equation $\Delta u + u = f$ has a smooth solution of class $C^\infty(\overline{\Omega})$ satisfying the Dirichlet condition $u = g$ on $\partial\Omega$.

2. (20 points) Let $\alpha \in (0,1)$ and $a_{ij}, b_i, c, f \in C^\alpha(\overline{B}_2^+)$ for $i,j \in \{1, \cdots, n\}$. Also, $a_{ij}(x) = a_{ji}(x)$ holds in $\overline{B}_2^+$ and

$$\|a_{ij}\|_{C^\alpha(\overline{B}_2^+)}, \|b_i\|_{C^\alpha(\overline{B}_2^+)}, \|c_{ij}\|_{C^\alpha(\overline{B}_2^+)} \leq \Lambda. \tag{2}$$

Moreover, there exists $\lambda > 0$ such that $\lambda\|\xi\|^2 \leq a_{ij}(x)\xi_i\xi_j$ holds for $x \in \overline{B}_2^+$ and $\xi \in \mathbb{R}^n$. Then, show that there exists some constant $C = C(n, \alpha, \lambda, \Lambda)$ such that

$$\|u\|_{C^{2,\alpha}(\overline{B}_1^+)} \leq C\left(\|f\|_{C^\alpha(\overline{B}_2^+)} + \|u\|_{C^0(\overline{B}_2^+)}\right), \tag{3}$$

holds for every $u \in C^{2,\alpha}(\overline{B}_2^+)$ satisfying $f = a_{ij}u_{ij} + b_iu_i + cu$ in $\overline{B}_2^+$.

3. (10 points) Let Ω be a bounded open set in $\mathbb{R}^n$ having the uniform exterior sphere boundary condition. Suppose that a_{ij}, b_i, c, f $(i,j \in \{1, \cdots, n\})$ are bounded functions defined over Ω satisfying $c(x) \leq 0$, $|a_{ij}(x)| \leq \Lambda$, $|b_i(x)| \leq \Lambda$ in Ω. Moreover,

there exists $\lambda > 0$ such that $\lambda\|\xi\|^2 \le a_{ij}(x)\xi_i\xi_j$ holds for $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Then, show that there exists some constant $C = C(n, \Omega, \lambda, \Lambda)$ such that

$$\sup_{\Omega} |u| \le C \sup_{\Omega} |f|, \tag{4}$$

holds for every $u \in D^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $f = a_{ij}u_{ij} + b_i u_i + cu$ in Ω and $u = 0$ on $\partial\Omega$.

4. (10 points) Suppose that a smooth function $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ satisfies $u_{ttt} - 2u_{ttx} - u_{txx} + 2u_{xxx} = 0$ at all $(x, t) \in \mathbb{R} \times [0, T)$. Moreover, $u(x, 0) = g(x) = e^x$, $u_t(x, 0) = h(x) = x + 2e^x$, and $u_{tt}(x, 0) = k(x) = 4e^x + 2$ hold for all $x \in \mathbb{R}$. Find all possible solutions.

Hint: $\partial_t^3 - 2\partial_t^2\partial_x - \partial_t\partial_x^2 + 2\partial_x^3 = (\partial_t - 2\partial_x)(\partial_t - \partial_x)(\partial_t + \partial_x)$.

5. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary $\partial\Omega$. Let $\{(w_i, \lambda_i)\}_{i=1}^{\infty} \subset C_0^{\infty}(\Omega) \times \mathbb{R}$ be the sequence pairs of the Dirichlet Laplace eigenfunction and eigenvalue satisfying $\|w_i\|_{L^2(\Omega)} = 1$, $0 < \lambda_i \le \lambda_{i+1}$, $\lim_{i\to+\infty} \lambda_i = +\infty$, $\langle w_i, w_j\rangle_{L^2(\Omega)} = \delta_{ij}$, and $\{w_i\}_{i=1}^{\infty}$ spans $L^2(\Omega)$.

Suppose that a smooth function $u \in C^{\infty}\big(\overline{\Omega} \times [0, T)\big)$ satisfies the *damped wave equation*

$$u_{tt} + u_t = \Delta u - u \tag{5}$$

in $\Omega \times [0, T)$ and the Dirichlet condition $u = 0$ on $\partial\Omega \times [0, T)$.

(A) (3 points) Show that the smooth function $a_i(t) = \langle u(x, t), w_i(x)\rangle_{L^2(\Omega)}$ satisfies

$$a_i'' + a_i' + (\lambda_i + 1)a_i = 0. \tag{6}$$

(B) (2 points) The ODE theory implies that the solution $a_i(t)$ to (6) must be

$$a_i(t) = \alpha_i e^{-\frac{t}{2}} \cos(\mu_i t) + \beta_i e^{-\frac{t}{2}} \sin(\mu_i t) \tag{7}$$

for some constants $\alpha_i, \beta_i \in \mathbb{R}$, where $\mu_i = \sqrt{\lambda_i + \frac{3}{4}}$.
Determine α_i and β_i in terms of $g(x) = u(x, 0)$, $h(x) = u_t(x, 0)$, $w_i(x)$, and μ_i.

(C) (5 points) Show that $\|u\|_{H^1(\Omega)} \le Ce^{-\frac{t}{2}}$ for some constant C depending on g, h and their derivatives.

6. Suppose that a smooth function $u \in C^{\infty}\big(\mathbb{R}^n \times [0, T)\big)$ satisfies the *damped wave equation*

$$u_{tt} + u_t = \Delta u - u \tag{8}$$

in $\mathbb{R}^n \times [0, T)$.

(A) (10 points) Show that the following energy is non-increasing

$$E(t) = \frac{1}{2}\int_{B(R-t;x_0)} |\nabla u|^2 + |u_t|^2 + u^2 dx \tag{9}$$

where $B(R-t;x_0) = \{x \in \mathbb{R}^n : |x - x_0| \le R - t\}$, $R \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$.

(B) (5 points) Suppose that the initial data $g(x) = u(x,0)$ and $h(x) = u_t(x,0)$ are compactly supported. Show that $u(x,t)$ is also compactly supported for each $t \ge 0$.

(C) (10 points) Suppose that the initial data $g(x) = u(x,0)$ and $h(x) = u_t(x,0)$ are compactly supported. We define the energy $J(t)$ by

$$J(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + |u_t|^2 + u^2 dx + \frac{1}{10}\int_{\mathbb{R}^n} u u_t \, dx. \tag{10}$$

Show that $J(t) \ge \frac{1}{10}\|u\|^2_{H^1(\mathbb{R}^n)}$ and $J' + \frac{1}{10}J \le 0$. Verify $\|u\|_{H^1(\mathbb{R}^n)} \le Ce^{-\frac{t}{20}}$ for some constant C.

7. Given a function $g \in C^\infty([0,\pi])$ with $g(0) = g(\pi) = 0$, we denote by $X_g \subset L^\infty(\Omega)$ the set of smooth *uniformly bounded* functions $u(x,y) = u(r\cos\theta, r\sin\theta)$ satisfying

$$0 = \Delta u + 2|x|^{-2}u = \partial^2_{rr}u + \frac{\partial_r u}{r} + \frac{\partial_{\theta\theta}u}{r^2} + \frac{2u}{r^2} \tag{11}$$

in $\Omega = \{(r\cos\theta, r\sin\theta) : 0 \le \theta \le \pi, r \ge 1\} \subset \mathbb{R}^2$, and satisfying the boundary condition

$$u(\cos\theta, \sin\theta) = g(\theta) \quad \text{for} \quad \theta \in [0,\pi], \quad u(r,0) = u(-r,0) = 0 \quad \text{for} \quad r \ge 1. \tag{12}$$

Given $u \in X_g$ and $m \in \mathbb{N}$, we define a smooth function $a_m \in C^\infty([1,\infty))$.

$$a_m(r) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\int_0^\pi u(r\cos\theta, r\sin\theta)\sin(m\theta)d\theta. \tag{13}$$

We know that $\{2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\sin(m\theta)\}_{m=1}^\infty$ form an orthogonal basis of $L^2((0,\pi))$. Thus,

$$u(r\cos\theta, r\sin\theta) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\sum_{m=1}^\infty a_m(r)\sin(m\theta). \tag{14}$$

(A) (2 points) Show that a_m satisfies $|a_m| \le C$ for some constant C and the following equation

$$a_m'' + r^{-1}a_m' + r^{-2}(2 - m^2)a_m = 0. \tag{15}$$

(B) (6 points) The ODE theory implies that the solutions to (15) must be

$$a_1(r) = \alpha_1\cos(\log r) + \beta_1\sin(\log r), \tag{16}$$

for some constants $\alpha_1, \beta_1 \in \mathbb{R}$. Moreover, for each $k \ge 2$

$$a_k(r) = \alpha_k r^{-\sqrt{k^2-2}} + \beta_k r^{\sqrt{k^2-2}}. \tag{17}$$

for some constants $\alpha_k, \beta_k \in \mathbb{R}$.

Determine α_m, β_m except β_1. What are the possible β_1?

(C) (7 points) Let $X_g^0 \subset X_g$ consist of the solutions u which converges to 0 as $r \to +\infty$. What are the possible sizes of the set X_g^0? Provide the conditions of g determining the size of X_g^0.

8. Ω is a smooth bounded open domain in $\mathbb{R}^n$. We would like to solve the semi-linear elliptic equation

$$\Delta u = u^3 \qquad \text{in } \Omega, \tag{18}$$

for the Dirichlet condition $u = g$ on $\partial\Omega$, where $g \in C^\infty(\overline{\Omega})$ and $\|g\|_{L^\infty} = \epsilon$ is small.

(A) (3 points) Briefly verify that there exists a unique harmonic function $v_1 \in C^\infty(\overline{\Omega})$ such that $v_1 = g$ on $\partial\Omega$. Moreover, (by using the maximum principle) show that

$$\sup_\Omega |v_1| \leq \sup_{\partial\Omega} |g|. \tag{19}$$

(B) (7 points) Briefly verify that given $v, f \in C^\infty(\overline{\Omega})$ the linear equation $\Delta w - 3v^2 w = f$ has a unique solution $w \in C^\infty(\Omega)$ satisfying $w = 0$ on $\partial\Omega$. Moreover, (by using the comparison principle and barriers) show that

$$\sup_\Omega |w| \leq M \sup_\Omega |f|, \tag{20}$$

for some M depending on n, Ω.

HINT: Use a paraboloid as a barrier.

(C) (3 points) Let $v_2 \in C^\infty(\overline{\Omega})$ be the solution to $\Delta v_2 - 3v_1^2 v_2 = f_2 = v_1^3$ satisfying $v_2 = 0$ on $\partial\Omega$. Show that there exists small ϵ such that

$$\sup_\Omega |v_2| \leq M \sup_\Omega |v_1|^3 \leq \epsilon^2. \tag{21}$$

(D) (4 points) For $k \geq 3$, we let $v_{k+1} \in C^\infty(\overline{\Omega})$ be the solution to

$$\Delta v_{k+1} - 3\Big(\sum_{m=1}^{k} v_m\Big)^2 v_{k+1} = f_{k+1} = 3\Big(\sum_{m=1}^{k-1} v_m\Big) v_k^2 + v_k^3 = \Big(\sum_{m=1}^{k} v_m\Big)^3 - \sum_{m=1}^{k} \Delta v_m, \tag{22}$$

satisfying $v_{k+1} = 0$ on $\partial\Omega$. Show that there exists small ϵ such that

$$\sup_\Omega |v_{k+1}| \leq \epsilon^{k+1}. \tag{23}$$

(E) (3 points) Let $u_k = \sum_{m=1}^{k} v_m$ and $\bar{u} = \lim_{k\to+\infty} u_k$. Show that

$$\lim_{k\to\infty} \sup_{\Omega} \left| \Delta u_k - \bar{u}^3 \right| = 0. \tag{24}$$

9. Suppose that $u : \mathbb{R}^n \times [0, +\infty)$ is a smooth function such that $u(x, t) = u(x+e_i, t)$ holds for every $i \in \{1, \cdots, n\}$ and the following equation holds

$$u_t = \Delta u - \sum_{i,j} \frac{u_i u_j u_{ij}}{1 + |\nabla u|^2}. \tag{25}$$

(A) (5 points) Show that the following holds for $t \geq 0$.

$$|\nabla u(x, t)|^2 \leq \sup_{x\in\mathbb{R}^n} |\nabla u(x, 0)|^2. \tag{26}$$

HINT: Maximum principle.

(B) (5 points) Show that the following holds for $t \geq 0$.

$$\frac{d}{dt} \int_{\Omega} \sqrt{1 + |\nabla u(x, t)|^2}\, dx \leq 0, \tag{27}$$

where $\Omega = (0, 1)^n \subset \mathbb{R}^n$.

10. (10 points) Let Ω be a convex bounded open set in $\mathbb{R}^n$ with smooth boundary. Suppose that $u \in C^\infty(\Omega \times [0, +\infty))$ satisfies $u_t = \Delta u$ in $\overline{\Omega} \times [0, +\infty)$ and $u = g$ on $\partial\Omega \times [0, +\infty)$, where $g \in C^\infty(\overline{\Omega})$. Let $w : \overline{\Omega} \to \mathbb{R}$ be the harmonic function satisfying $w = g$ on $\partial\Omega$. Show that

$$\lim_{t\to+\infty} \sup_{x\in\Omega} |u(x, t) - w(x)| = 0. \tag{28}$$

HINT: Problem set 1.